4.1

Linear Inequations in One Variable

Learning Objectives:

- To define an inequation and linear inequation
- To solve the linear inequations in one variable and to find their solution sets
- To represent the solution set of an inequation on the Real Line

AND

- To practice the related problems

Inequations

If $x \neq 5$ then either $x < 5$ or $x > 5$.

The symbols $' > '$ and $' < '$ are known as symbols of inequalities and the statements involving inequalities $x < 5$, $x > 5$ are known as inequations.

For example, the sum of the measures of two sides of a triangle is greater than the measure of its third side. This is an inequation. If a, b, c denote the measures of the sides of a triangle, then $a + b > c$

If a person has driving license, his age must be at least 18. If x denotes the age of the person, then $x \geq 18$.

In an inequation, the algebraic expressions are related by one of the four symbols of inequalities:

>	Greater than
≥	Greater than or equal to
<	Less than
≤	Less than or equal to

*A mathematical expression containing one or more of the symbols $<, \leq, >, \geq$ is called an **inequation**. A linear inequation is a statement of inequality in which the highest power of the variable is one.*

Examples of inequations in one variable are

$$2x + 3 \leq 5\ ,\ 4x > 3x - 5$$
$$-3 \leq -x + 5\ ,\ 2x + 3 \geq 0$$

In general, a linear inequation can be written as

$$ax + b < 0, \qquad ax + b \leq 0, \qquad ax + b > 0, \qquad ax + b \geq 0$$

where $a, b \in \boldsymbol{R}$ *and* $a \neq 0$. *The direction of the inequality symbol is called the sense of the inequality.*

Solving Inequations

A simple linear equation has only one root (solution), but a linear inequation may have more than one root.

For example, the root of $3x - 3 = 18$ is 7, but for $3x - 3 \geq 18$ the roots are all real numbers greater than or equal to 7. Solving an inequation involves finding the set of values for the variable which satisfy the inequality.

The *solution set* of an inequation is the set of numbers that when substituted in place of the variable makes the inequation a true statement. The solution set is also known as the *truth set*.

To solve an inequation, we must isolate the variable on one side of the inequality symbol. *To isolate the variable, we use the same basic techniques used in solving equations*. The following properties hold good for inequations. The first two properties are the same as in the case of solving equations.

1. *The same quantity may be added to or subtracted from both sides of an inequation.*
2. *Both sides of an inequation may be multiplied or divided by the same **positive** quantity.*
3. *On multiplying or dividing both sides of an inequation by the same **negative** quantity, the sign of inequality is reversed.*

The solution of the inequation remains the same after these operations.

For example, the inequality $x > 5$ is true for $x = 6$. If we multiply both sides by 3, we have

$$3x > 15 \text{ because } 18 > 15$$

On the other hand, if we multiply both sides by −3, we have, $-3x < -15$ because $-18 < -15$.

Example 1: Solve the inequation $2x + 6 < 12$.

Solution:

$$2x + 6 < 12 \Rightarrow 2x + 6 - 6 < 12 - 6 \Rightarrow 2x < 6 \Rightarrow \frac{2x}{2} < \frac{6}{2} \Rightarrow x < 3$$

Any number less than 3 satisfies the inequation.

4.2

Inequations Involving Absolute Values

Learning Objectives:

- To study the inequations involving absolute values

  AND

- To solve related problems

Inequations Involving Absolute Values

The inequation $|a| < D$ says that the distance from a to 0 is less than D. Therefore a must lie between D and $-D$. If D is any positive number, then

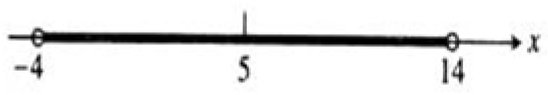
$$|a| < D \quad \Leftrightarrow \quad -D < a < D$$
$$|a| \le D \quad \Leftrightarrow \quad -D \le a \le D$$

Example 1: Solve the inequation $|x-5| < 9$ and graph the solution set on the real line.

Solution:

$$|x-5| < 9 \Rightarrow -9 < x-5 < 9$$
$$\Rightarrow -9+5 < x < 9+5 \Rightarrow -4 < x < 14$$

The solution is the open interval $(-4,14)$



Example 2: Solve the inequation $\left|5-\frac{2}{x}\right| < 1$

Solution:

$$\left|5-\frac{2}{x}\right| < 1 \Rightarrow -1 < 5-\frac{2}{x} < 1,\ x \ne 0$$
$$\Rightarrow -6 < -\frac{2}{x} < -4 \Rightarrow 3 > \frac{1}{x} > 2$$
$$\Rightarrow \frac{1}{3} < x < \frac{1}{2}$$

The solution set is the open interval $\left(\frac{1}{3},\frac{1}{2}\right)$

Example 3: Solve the inequations and graph the solution set:

$a)$ $\quad |2x-3| \le 1$

$b)$ $\quad |2x-3| > 1$

Solution:

$a)$ $\quad |2x-3| \le 1 \Rightarrow -1 \le 2x-3 \le 1$
$$\Rightarrow 2 \le 2x \le 4 \Rightarrow 1 \le x \le 2$$

The solution set is the closed interval $[1,2]$.

$b)$ $\quad |2x-3| > 1 \Rightarrow 2x-3 > 1 \quad or \quad -(2x-3) > 1$
$$\Rightarrow 2x-3 > 1 \quad or \quad 2x-3 < -1$$
$$\Rightarrow x-\frac{3}{2} > \frac{1}{2} \quad or \quad x-\frac{3}{2} < -\frac{1}{2}$$
$$\Rightarrow x > 2 \quad or \quad x < 1$$

The solution set is $(-\infty,1) \cup (2,\infty)$

(a)

(b)

Inequations of the form $|x-a| < k$ and $|x-a| > k$ arise often, and their interpretation is as follows.

Inequation ($k > 0$)	*Geometric Interpretation*	*Alternative forms of the Inequation*
$\lvert x-a \rvert < k$	x is within k units of a.	$-k < x-a < k$ $a-k < x < a+k$ $x \in (a-k, a+k)$
$\lvert x-a \rvert > k$	x is more than k units away from a.	$x-a < -k$ or $x-a > k$ $x < a-k$ or $x > a+k$ $x \in (-\infty, a-k) \cup (a+k, \infty)$

4.3

Systems of Linear Inequations

Learning Objectives:

- To study the linear inequation in two variables and to represent the solution set graphically

  AND

- To solve a system of linear inequations graphically and to find the solution region

Linear Inequations in Two Variables

A linear inequation in two variables is any expression that can be put in the form

$$ax + by < c$$

where a, b, and c are real numbers (a and b not both zero). The inequality symbol can be any one of the following four: $<, \le, >, \ge$.

Some examples of linear inequations are

$$2x+3y<6 \quad y\ge 2x+1 \quad x-y\le 0$$

Although not all of these examples have the form $ax + by < c$, each one can be put in that form.

The solution set for a linear inequation in two variables is a section of the coordinate plane. The ***boundary*** for the section is found by replacing the inequality symbol with an equal sign and graphing the resulting equation. The ***boundary is included*** in the solution set and is represented by a ***solid line***, if the inequality symbol is $\le$ or $\ge$. The ***boundary is not included***, indicated by a ***broken line***, if the symbol is $<$ or $>$.

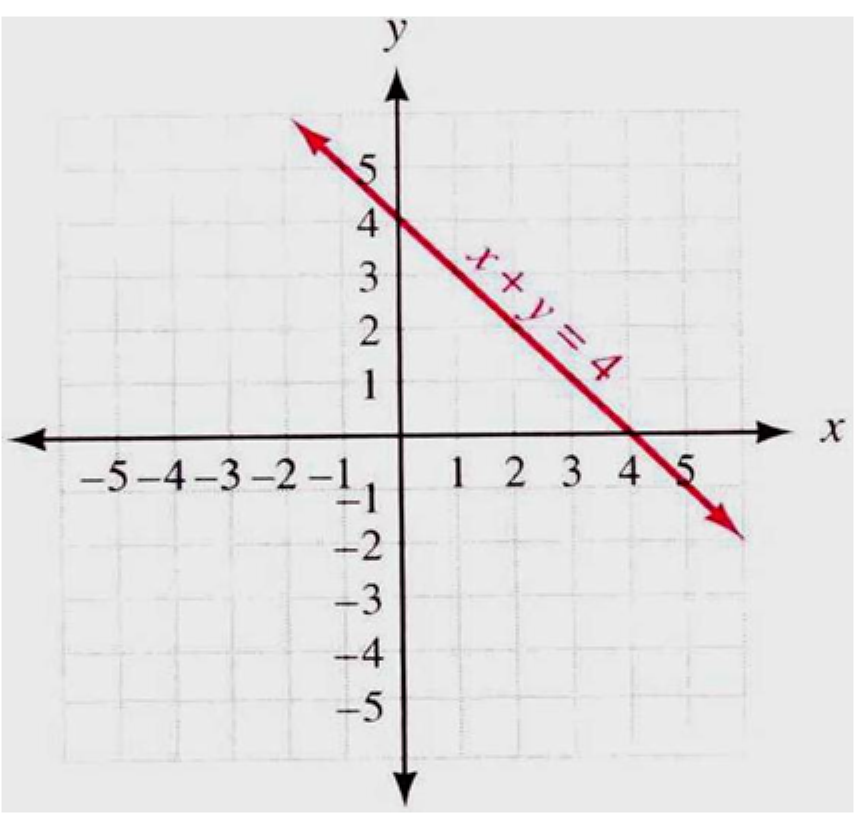
Example 1

Graph the solution set for $x + y \le 4$.

Solution

The boundary for the required graph is the graph of $x + y = 4$. The boundary is included in the solution set because the inequality symbol is $\le$.

Figure below is the graph of the boundary.



The boundary separates the coordinate plane into two regions: the region above the boundary and the region below it. The solution set for $x + y \le 4$ is one of these two regions along with the boundary.

To find the correct region, we simply choose any convenient point that is not on the boundary. We then substitute the coordinates of the point into the given inequation $x + y \le 4$. If the point we have chosen satisfies the inequation, then it is a member of the solution set, and we can assume that all points on the same side of the boundary as the chosen point are also in the solution set.

If the coordinates of our chosen point do not satisfy the given inequality, then the solution set lies on the other side of the boundary.

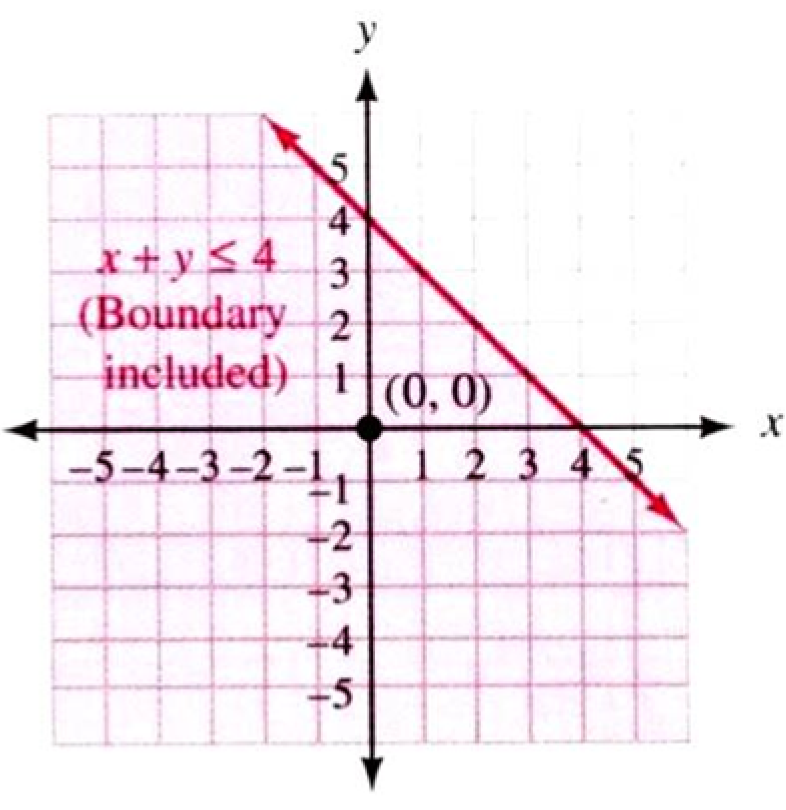
In this example, a convenient point that is not on the boundary is the origin. Substituting $(0,0)$ into the given inequality gives us

$$0+0\le 4$$

which is a true statement.

Because the origin is a solution to the inequality $x + y \le 4$ and the origin is below the boundary, all other points below the boundary are also solutions.

Figure below is the graph of $x + y \le 4$.



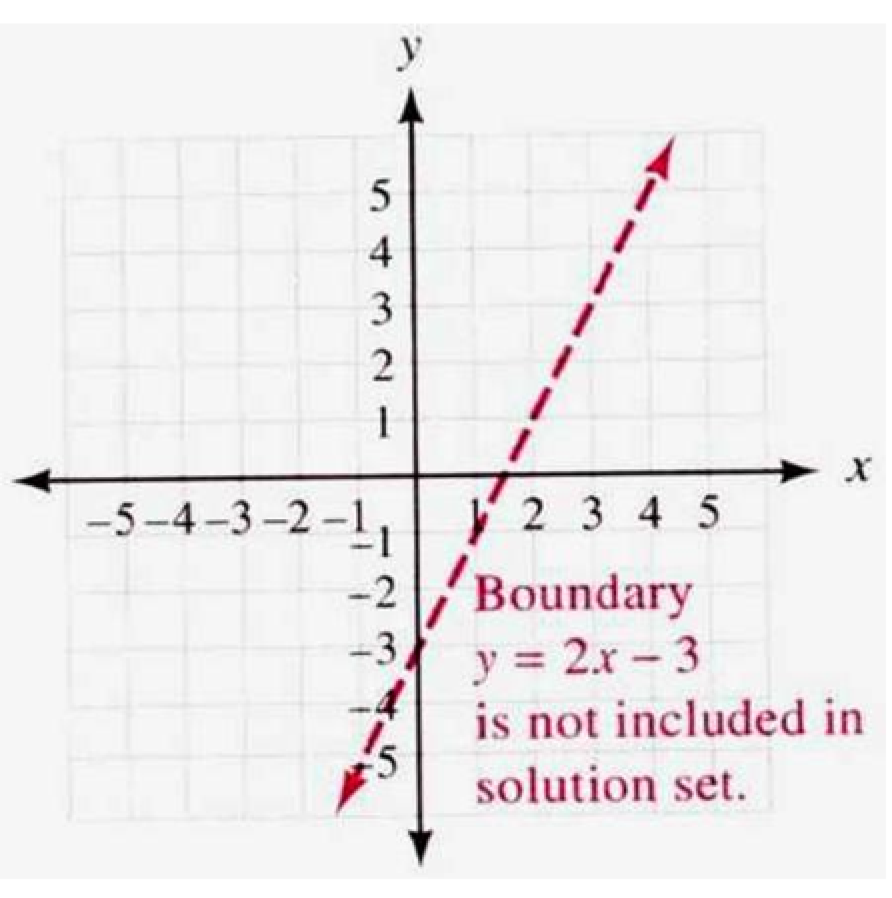
We note that the region above the boundary is described by the inequality $x + y \ge 4$.

Example 2

Graph the solution set for $y < 2x - 3$.

Solution

The boundary is the graph of $y = 2x - 3$, a line with slope 2 and y-intercept -3. The boundary is not included because the inequality symbol is $<$. We therefore use a broken line to represent the boundary.



We choose the origin as test point which results in

$$0 < 2(0)-3$$
$$\text{i.e., } 0 < -3$$

a false statement. Therefore the solution set must lie on the non-origin side of the boundary.